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The field of rational numbers is algebraically closed in some of its topological completions[1]

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Abstract

Some examples of locally unbounded topologizations of the field of rational numbers are given. Their completions are fields in which the rational number field is algebraically closed. A similar development is given for the ring of rational integers and other domains. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

All fields and rings considered in this paper are assumed to be commutative. We recall that a subset S of a commutative topological ring R is bounded if given any neighborhood V of zero, there exists a neighborhood U of zero such that $SU \subseteq V$. If R is a field, this is equivalent to saying that given any neighborhood V of zero, there exists a nonzero element $x \in R$ such that $Sx \subseteq V$ (see [7, 10, Theorem 3, p. 42 or 14, Lemma 12, p. 26]). A ring topology on R is *locally bounded* if there is a bounded neighborhood of zero. A nonzero element a of a topological ring is called topologically nilpotent if $\lim_{n\to\infty} a^n = 0$. A field with a ring topology is called *completable* or full if its completion is a field.

In 1964 Hinrichs [2] constructed, using an inductive procedure, ring topologies on the ring of integers $\mathbb{Z}$ which do not have any basis of zero neighborhoods consisting of ideals. In 1968, with a similar method, Mutylin [6] constructed completable locally

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unbounded topologizations of the rational number field $\mathbb{Q}$. In this article we construct, in an explicit way, some locally unbounded field topologies on the rational field. Its completions are fields in which $\mathbb{Q}$ is algebraically closed. We also show ring topologies on the ring of integers with similar features. An analogous development can be given for ring and field topologies in other domains and fields, as we suggest in Section 5, with the ring of polynomials $k[X]$ and its quotient field $k(X)$.

We obtained first the topologies and the completion fields introduced here by means of nonstandard analysis. For an introduction to this subject see [3, 11]. We follow a standard treatment in order to get a more accessible exposition and a broader audience; but doing that, we have to add some technical lemmas and the proofs become more cumbersome.

Throughout this article parameters a, b, c, d with subscripts will denote rational integers and $(p_n)_{n\in\mathbb{N}}$ will denote a sequence of natural numbers which satisfies the following properties:

$$p_1 = 1, \;\; p_2 \geq 2 \quad \text{and for all } n \in \mathbb{N}, \;\; p_{n+1} \geq (p_n)^n. \tag{1}$$

By induction, $p_{n+1} \geq 2^{n!}$ for $n \in \mathbb{N}$. Moreover, $p_{n+1} \geq n^n$ for $n \geq 3$.

2. Some ring topologies on $\mathbb{Z}$

We are going to define some ring topologies on the rational integers $\mathbb{Z}$. We show some algebraic properties of the corresponding completions in Theorems 3 and 4. For this purpose we take a sequence of natural numbers $(p_n)_{n\in\mathbb{N}}$ which satisfies (1). For each $m \in \mathbb{N}$, we define the following subset of $\mathbb{Z}$:

$$V_m = \left\{ \sum_{n\geq m}^{l} a_n p_n : a_n \in \mathbb{Z} \text{ and } |a_n|^m < p_{n+1} \right\}. \tag{2}$$

For $m \geq 3$, the given representation of the elements of V_m is unique. The following result provides us a unique representation of rational integers according to this representation of the elements of V_m.

Lemma 1. *Each element $a \in \mathbb{Z}$ can be expressed uniquely as $a = \sum_{n=1}^{s} a_n p_n$ with $a_n \in \mathbb{Z}$ satisfying for all $k \in \mathbb{N}$ the condition*

$$\left| \sum_{n=1}^{k} a_n p_n \right| \leq \frac{p_{k+1}}{2} \quad \textit{and, if equality holds, then } \sum_{n=1}^{k} a_n p_n > 0. \tag{3}$$

Proof. Let p_s be the smallest element of the sequence $(p_n)_{n\in\mathbb{N}}$ which satisfies $|a| \leq p_s$. There are integers a_s, r_s such that $a = a_s p_s + r_s$ with $|r_s| \leq p_s/2$ and, if equality holds,

then $r_s = p_s/2$ according to the following cases:
- if $p_s/2 < a \le p_s$, then $a_s = 1$ and $r_s = a - p_s$,
- if $-p_s/2 < a \le p_s/2$, then $a_s = 0$ and $r_s = a$,
- if $-p_s \le a \le -p_s/2$, then $a_s = -1$ and $r_s = a + p_s$.

Now, we proceed with r_s; we can express $r_s = a_{s-1}p_{s-1} + r_{s-1}$, where $|r_{s-1}| \le p_{s-1}/2$ and, if equality holds, then $r_{s-1} = p_{s-1}/2$. We continue till we get the expression $a = \sum_{n=1}^{s} a_n p_n$. Let us show the uniqueness by contradiction. Suppose that an integer is represented by two different sums satisfiying condition (3):

$$\sum_{n=1}^{s} a_n p_n = \sum_{n=1}^{m} b_n p_n.$$

If $s < m$ and $b_m \ne 0$, we can rearrange the above identity,

$$\left|\sum_{n=1}^{s} a_n p_n - \sum_{n=1}^{s} b_n p_n\right| = \left|\sum_{s+1}^{m} b_n p_n\right|.$$

Taking into account the bound in (3), we deduce that

$$\left|\sum_{n=1}^{s} a_n p_n - \sum_{n=1}^{s} b_n p_n\right| < p_{s+1}$$

and

$$\left|\sum_{s+1}^{m} b_n p_n\right| \ge p_{s+1},$$

yielding a contradiction. If $s = m$, let j be the largest index at which $a_n \ne b_n$; we get the identity

$$\left|\sum_{n=1}^{j-1} a_n p_n - \sum_{n=1}^{j-1} b_n p_n\right| = |a_j - b_j| p_j,$$

and again, a contradiction. □

It is easy to check that, for $m \ge 3$, the representation of the elements of V_m given in (2) coincides with that of Lemma 1.

We recall that for a sequence $\{U_n\}_{n\in\mathbb{N}}$ of subsets of a commutative ring R to be a fundamental system of neighborhoods at zero for a Hausdorff ring topology $\mathcal{T}$ on R, it suffices that the following properties hold:

$$\text{For all } n \quad 0 \in U_n, \quad U_n = -U_n, \quad U_{n+1} \subseteq U_n. \tag{4}$$

$$\text{For all } n \text{ there exists } k \text{ such that } U_k + U_k \subseteq U_n. \tag{5}$$

$$\text{For all } n \text{ there exists } k \text{ such that } U_k U_k \subseteq U_n. \tag{6}$$

For all n and $x \in R$ there exists k such that $xU_k \subseteq U_n$. (7)

$$\bigcap_{n\in\mathbb{N}} U_n = \{0\}. \tag{8}$$

If, in addition, R is a field, then $\mathscr{T}$ is a field topology if $\{U_n\}_{n\in\mathbb{N}}$ also satisfies the following condition:

For all n there exists k such that $(1+U_k)^{-1} \subseteq 1+U_n$. (9)

See [4, 5, 10, 14] for instance. It is left to the reader to check that the subsets V_m defined in (2) satisfy properties (4)–(8) (cf. [5, Paragraph 1]). In order to prove property (6), the product of two elements in the neighborhood V_k,

$$\alpha = \sum_{n=k}^{l} a_n p_n, \quad \beta = \sum_{n=k}^{l} b_n p_n,$$

must be written grouping the coefficients in the following way

$$\alpha\beta = (a_k b_k p_k) p_k + \sum_{m=k+1}^{l} \left(\left(\sum_{n=k}^{m} a_n p_n \right) b_m + a_m \left(\sum_{n=k}^{m-1} b_n p_n \right) \right) p_m. \tag{10}$$

Thus, we deduce the following result.

Theorem 1. *The family $\mathscr{B} = \{V_m\}_{m\in\mathbb{N}}$ is a fundamental system of neighborhoods of zero for a Hausdorff ring topology on $\mathbb{Z}$.*

We shall denote this ring topology by $\mathscr{T}$. This topology does not have an ideal base of zero neighborhoods and, therefore, it is locally unbounded (see [5, 13]).

The topological ring $(\mathbb{Z}, \mathscr{T})$ is not complete. We shall construct its completion $\widehat{\mathbb{Z}}$, which is, as usual, the quotient ring of the ring of Cauchy sequences by the ideal of all sequences converging to zero. We say that two Cauchy sequences are equivalent if they represent the same element in this quotient ring. In the sequel, $\sum_{n=1}^{\infty} a_n p_n$ will mean the element in $\widehat{\mathbb{Z}}$ which is the limit of the Cauchy sequence $(\sum_{n=1}^{m} a_n p_n)_{m\in\mathbb{N}}$, in case it exists. In order to describe this completion in an easier way, we need the next lemma.

Lemma 2. *Let $(\alpha_m)_{m\in\mathbb{N}}$ be a Cauchy sequence in $(\mathbb{Z}, \mathscr{T})$. There exists a sequence $(t_n)_{n\in\mathbb{N}}$ and a Cauchy sequence $(\omega_m)_{m\in\mathbb{N}}$ which is equivalent to $(\alpha_m)_{m\in\mathbb{N}}$ and satisfies $\omega_m = \sum_{n=1}^{m} t_n p_n$, for all m. In addition, $|t_n|^7 < p_{n+1}$, for all $n \geq 7$.*

Proof. There exists a subsequence $(\gamma_m)_{m\in\mathbb{N}}$ of the sequence $(\alpha_m)_{m\in\mathbb{N}}$ which satisfies $\gamma_{m_1} - \gamma_{m_2} \in V_{m+1}$, for all $m_1, m_2 \geq m$. The representation of the elements in V_m in (2) is unique for all $m \geq 3$. Besides, for all $m \geq 6$, the difference $\gamma_m - \gamma_6$ belongs to V_7. Thus we can write

$$\gamma_m = \gamma_6 + \sum_{n=7}^{\mu_m} t_{mn} p_n, \tag{11}$$

with $|t_{mn}|^7 < p_{n+1}$ and the index $\mu_m \geq m$. We set $t_1 = \gamma_6$ and we define

$$\omega_m = \begin{cases} \gamma_6, & \text{if } m \leq 6, \\ \gamma_6 + \sum_{n=7}^{m} t_{m\,n}\, p_m, & \text{otherwise.} \end{cases}$$

Let us show that $(\omega_m)_{m\in\mathbb{N}}$ is a Cauchy sequence equivalent to $(\gamma)_{m\in\mathbb{N}}$. Given a neighborhood of zero V_{m+1} in the basis $\mathscr{B}$ with $m \geq 6$, let us show that $\gamma_k - \omega_k \in V_{m+1}$, for all $k > \mu_m$. Since $\gamma_k - \gamma_m \in V_{m+1}$, we can write

$$\gamma_k - \gamma_m = \sum_{n=m+1}^{\kappa} u_n\, p_n$$

with $|u_n|^{m+1} < p_{n+1}$, for all $n \geq m+1$. Then

$$\begin{aligned} \gamma_k &= \gamma_6 + \sum_{n=7}^{\mu_m} t_{mn} p_n + \sum_{n=m+1}^{\kappa} u_n\, p_n \\ &= \gamma_6 + \sum_{n=7}^{m} t_{mn} p_n + \sum_{n=m+1}^{\mu_m} (t_{m\,n} + u_n)\, p_n + \sum_{n=\mu_m+1}^{\kappa} u_n p_n. \end{aligned}$$

(We do the case $\mu_m > m$; if $\mu_m = m$, the proof follows similarly.) Since

$$\sum_{n=7}^{\mu_m} t_{mn} p_n + \sum_{n=m+1}^{\kappa} u_n p_n \in V_7 + V_{m+1} \subseteq V_3,$$

the uniqueness of the representation of the elements in V_3 implies that

$$\begin{aligned} &t_{mn} = t_{kn} && \text{for all } n \in \{7, \dots, m\}, \\ &t_{mn} + u_n = t_{kn} && \text{for all } n \in \{m+1, \dots, \mu_m\}, \\ &u_n = t_{kn} && \text{for all } n \in \{\mu_m + 1, \dots, \kappa\}. \end{aligned}$$

Therefore, assuming $\kappa > k$,

$$\gamma_k - \omega_k = \sum_{n=k+1}^{\kappa} u_n p_n \in V_{m+1}.$$

This concludes the proof. □

In the sequel, if $a_n = 0$, we shall consider, for convenience, that $\log |a_n| = 0$.

Theorem 2. *The completion ring of* $\mathbb{Z}$ *with respect to the ring topology* $\mathscr{T}$ *can be described as follows*:

$$\widehat{\mathbb{Z}} = \left\{ \sum_{n=1}^{+\infty} a_n\, p_n : a_n \in \mathbb{Z} \text{ and } \lim_{n\to\infty} \frac{\log |a_n|}{\log (p_{n+1})} = 0 \right\}.$$

Proof. Given $a \in \widehat{\mathbb{Z}}$, there exists a Cauchy sequence (ω_m) which represents a and satisfies the conditions of the previous lemma. Then $a = \sum_{n=1}^{\infty} t_n p_n$. For all $k \geq 7$, there exists $m_k \geq 7$ such that $\omega_{m_2} - \omega_{m_1} = \sum_{n=m_1}^{m_2} t_n p_n \in V_k$, for all $m_2 \geq m_1 \geq m_k$. Since $|t_n|^7 < p_{n+1}$, by the uniqueness of the representation of the elements in V_k for $k \geq 3$, we conclude that $|t_n|^k < p_{n+1}$ and so $(\log|t_n|)/[\log(p_{n+1})] < 1/k$, for all $n \geq m_k$. Therefore, the corresponding limit is zero. The converse is established similarly. □

Given an element in $\widehat{\mathbb{Z}}$, presented as in the theorem above, it is easy to check that, for all $k \in \mathbb{N}$, there exists m_k such that

$$\left|\sum_{n=1}^{m} a_n p_n\right|^k < p_{m+1} \quad \text{for all } m \geq m_k. \tag{12}$$

Notice that if an element of $\widehat{\mathbb{Z}}$ is written in two different ways

$$\sum_{n=1}^{\infty} a_n p_n = \sum_{n=1}^{\infty} b_n p_n,$$

there is an $n_0 \in \mathbb{N}$ such that the cofficients a_n and b_n are equal for all $n \geq n_0$. In addition, we have a unique representation of the elements of $\widehat{\mathbb{Z}}$ in the following sense.

Lemma 3. *Every element $\alpha = \sum_{n=1}^{\infty} a_n p_n \in \widehat{\mathbb{Z}}$ is uniquely represented in the form described in Theorem 2 with the coefficients a_n satisfying the bounds* (3).

Proof. Applying (12), there exists $m_2 \geq 2$ such that

$$\left|\sum_{n=1}^{m} a_n p_n\right|^2 < p_{m+1} \quad \text{for all } m \geq m_2.$$

Consequently, $|\sum_{n=1}^{m} a_n p_n| < p_{m+1}/2$ for all $m \geq m_2$. Let $b = \sum_{n=1}^{m_2} a_n p_n \in \mathbb{Z}$; applying Lemma 1, we can express $b = \sum_{n=1}^{m_2} b_n p_n$ where the coefficients b_n fulfill (3) for $k = 1, \ldots, m_2$. Thus $\alpha = \sum_{n=1}^{m_2} b_n p_n + \sum_{m_2+1}^{\infty} a_n p_n$ have all its coefficients satisfying the bounds (3). □

The next result informs us about the basic algebraic properties of the completion ring $\widehat{\mathbb{Z}}$.

Theorem 3. *The ring $\widehat{\mathbb{Z}}$ is an integral domain which satisfies the following properties:*
- *if $a, b \in \widehat{\mathbb{Z}} \backslash \mathbb{Z}$, then $ab \in \widehat{\mathbb{Z}} \backslash \mathbb{Z}$,*
- *the only units of $\widehat{\mathbb{Z}}$ are* 1 *and* -1,
- *for all primes $p \in \mathbb{N}$, p is irreducible in $\widehat{\mathbb{Z}}$.*

Proof. We can express $a = \sum_{n=1}^{\infty} a_n p_n$ and $b = \sum_{n=1}^{\infty} b_n p_n$ satisfying conditions of Theorem 2. There are infinitely many nonzero a_n and b_n. We also express their product $ab = \sum_{n=1}^{\infty} c_n p_n$ in the same fashion, where the coefficients c_n are as follows. Let

$c_1 = a_1 b_1$, for $m > 1$ we have

$$c_m = \left(\sum_{n=1}^{m-1} a_n p_n\right) b_m + \left(\sum_{n=1}^{m-1} b_n p_n\right) a_m + a_m b_m p_m.$$

We consider three cases. If $a_m = b_m = 0$, then $c_m = 0$. If $a_m \neq 0$ and $b_m = 0$, the coefficient c_m equals $a_m \sum_{n=1}^{m-1} b_n p_n$, which is nonzero for m big enough. If $a_m b_m \neq 0$, then for m big enough, we have

$$|a_m b_m p_m| > \left(\sum_{n=1}^{m-1} a_n p_n\right) b_m + \left(\sum_{n=1}^{m-1} b_n p_n\right) a_m.$$

It follows that $c_m \neq 0$, whence $ab \in \widehat{\mathbb{Z}} \setminus \mathbb{Z}$. The two other statements are immediate consequences. □

We can extend the topology $\mathcal{T}$ from $\mathbb{Z}$ to $\widehat{\mathbb{Z}}$; for convenience, we denote the extended topology also by $\mathcal{T}$. A fundamental system of neighborhoods at zero is $\widehat{\mathcal{B}} = \{\overline{V_m}\}_{m \geq 3}$, where $\overline{V_m}$ is the closure of $V_m \in \mathcal{B}$ in the new topology on $\widehat{\mathbb{Z}}$ (see [10, Theorem 5, p. 175]). It is easily checked that the neighborhoods in the basis $\widehat{\mathcal{B}}$ are the following sets:

$$\overline{V_m} = \left\{\sum_{n=m}^{\infty} a_n p_n \in \widehat{\mathbb{Z}}\colon a_n \in \mathbb{Z},\ |a_n|^m < p_{n+1}\right\}.$$

In the rest of this section, given the element $\alpha = \sum_{n=1}^{\infty} a_n p_n \in \widehat{\mathbb{Z}}$, we shall call $\alpha_k = \sum_{n=k}^{\infty} a_n p_n$ and $\beta_k = \sum_{n=1}^{k-1} a_n p_n$. The next theorem shows a peculiarity of the completion ring $\widehat{\mathbb{Z}}$. We need the following technical lemma to prove it.

Lemma 4. *Given the element $\alpha = \sum_{n=1}^{\infty} a_n p_n \in \widehat{\mathbb{Z}}$ and a natural number $m \geq 2$, for all $t \in \mathbb{N}$, there exists a number $k_t \geq t$ such that, for all $k \geq k_t$, we have $(\alpha_k)^m = \sum_{n=k}^{\infty} c_{kn} p_n \in \overline{V_t}$, where $|c_{kn}|^t < p_{n+1}$, for all $n \geq k$.*

Proof. The power $(\alpha_k)^m$ can be written as

$$(\alpha_k)^m = \sum_{n=k}^{\infty} \left(\sum_{\substack{k \leq n_i \leq n \\ i=1,\dots,m-1}} \prod_{i=1}^{m-1} a_{n_i} p_{n_i} \right) a_n p_n = \sum_{n=k}^{\infty} c_{kn} p_n. \tag{13}$$

Notice that the sum within the brackets has $(n-k+1)^{m-1}$ terms. Let $b_n = \max\{|a_j|\colon k \leq j \leq n\}$. Given $t \in \mathbb{N}$, there is $k_t \geq 3mt$ such that for all $n \geq k_t$ the inequalities $(b_n)^{3mt} \leq p_{n+1}$ and $n^m \leq p_n$ hold. For $k \geq k_t$, we can now bound the coefficients c_{kn} in (13) as follows:

$$|c_{kn}| \leq n^{m-1} (b_n)^m (p_n)^{m-1} \leq (b_n)^m (p_n)^m \leq (p_{n+1})^{1/3t} (p_{n+1})^{1/3t} \leq (p_{n+1})^{1/t}. \qquad \square$$

Theorem 4. *The ring $\mathbb{Z}$ is algebraically closed in $\widehat{\mathbb{Z}}$.*

Proof. Let $\alpha = \sum_{n=1}^{\infty} a_n p_n \in \widehat{\mathbb{Z}} \backslash \mathbb{Z}$. Infinitely many a_n are nonzero. Let $R(X) \in \mathbb{Z}[X]$ be a polynomial of degree $m \geq 2$. We claim that $R(\alpha) \neq 0$. We split the polynomial $R(X)$ into the following sum:

$$R(X+Y) = P_0(X) + \sum_{i=1}^{m-1} P_i(X) Y^i + P_m Y^m,$$

where $P_0(X)$ is a polynomial of degree m, $\deg(P_i) < m$ for $i = 1, \ldots, m-1$, and P_m is a constant polynomial. For each $k > 1$ with $a_k \neq 0$, we write $\alpha = \beta_k + \alpha_k$, where β_k and α_k are as defined before Lemma 4. Therefore, $R(\alpha) = R(\beta_k + \alpha_k)$ splits into the following sum:

$$P_0(\beta_k) + \sum_{i=1}^{m} P_i(\beta_k)(\alpha_k)^i := P_0(\beta_k) + v. \tag{14}$$

We shall show that the two terms of (14) cannot have sum zero for k big enough. There exists $k_1 \in \mathbb{N}$ such that $P_0(\beta_k) \neq 0$ for all $k \geq k_1$. Given a natural number $t \geq 4$, there exists $k_t \in \mathbb{N}$ such that for all $k \geq k_t$ the inequalities $|P_i(\beta_k)|^t < p_k$, $(i = 0, \ldots, m-1)$ and $m^t < p_k$ hold. Besides, applying the previous lemma, there is $l_t \in \mathbb{N}$ such that for all $k \geq l_t$ and each $i = 1, \ldots, m$ we have $(\alpha_k)^i \in \overline{V}_{2t}$, i.e., $(\alpha_k)^i = \sum_{n=k}^{\infty} c_{in} p_n$, where $|c_{in}|^{2t} < p_{n+1}$ for all $n \geq k$. Therefore, for $k \geq \max\{t, k_1, k_t, l_t\}$, the second term of the sum (14) can be written as

$$\sum_{n=k}^{\infty} \left(\sum_{i=1}^{m} P_i(\beta_k) c_{in} \right) p_n := \sum_{n=k}^{\infty} b_n p_n.$$

In this expression the coefficients of the p_n are bounded as follows:

$$|b_n| = \left| \sum_{i=1}^{m} P_i(\beta_k) c_{in} \right| \leq m (p_k)^{1/t} (p_{n+1})^{1/2t} \leq (p_k)^{2/t} (p_{n+1})^{1/2t} \leq (p_{n+1})^{1/t}.$$

Therefore, $v = \sum_{n=k}^{\infty} b_n p_n \in \overline{V}_t$. We consider two cases. Firstly, if $v \notin \mathbb{Z}$, then $R(\alpha) = P_0(\beta_k) + v \neq 0$, since $P_0(\beta_k) \in \mathbb{Z}$. Secondly, if $v \in \mathbb{Z}$, then $v = \sum_{n=k}^{l} b_n p_n \in \overline{V}_t$. If $v = b_k p_k$, then $|v| \geq p_k$. If $l > k$, applying formula (3), then $|v| \geq p_l - p_l/2 > p_k$. On the other hand, $|P_0(\beta_k)| < (p_k)^{1/t} < p_k$. Consequently $|P_0(\beta_k)| < |v|$, and so $R(\alpha) = P_0(\beta_k) + v \neq 0$. □

Some examples of topological rings are obtained if we choose particular sequences $(p_n)_{n \in \mathbb{N}}$, let us consider two examples:

First, the natural numbers p_n being prime for all $n \geq 2$.

Second, $p_1 = 1$ and $p_n = p^{n!}$ for all $n \geq 2$, with p prime. In this case, the ring topology $\mathscr{T}$ is finer than the p-adic topology on $\mathbb{Z}$. Both completions are related as follows.

Lemma 5. *The completion $\widehat{\mathbb{Z}}$ is a subring of the ring of p-adic integers $\mathbb{Z}_p$.*

Proof. We consider the natural homomorphism from the completion of $(\mathbb{Z},\mathscr{T})$ into $\mathbb{Z}_p$. It suffices to check that the kernel of this continuous homomorphism is zero. Let $\alpha=\sum_{n=1}^{\infty} a_n p^{n!}$ be a nonzero element of the completion $(\widehat{\mathbb{Z}},\mathscr{T})$ presented as in Lemma 3. Let a_m be its lowest index nonzero coefficient. Now we consider this series to be summed in $\mathbb{Z}_p$. The new sum is the image of the original one under the natural homomorphism from the completion $(\widehat{\mathbb{Z}},\mathscr{T})$ to $\mathbb{Z}_p$. We show that the p-adic sum is not zero.

Observe that, for $a,b,t\in\mathbb{Z}$, $0<|b|<p^t$, implies $|b|_p>|p^t|_p\geq|ap^t|_p$, where $|\ |_p$ denotes the p-adic absolute value. Using this remark and (3), we obtain inductively that, for $k\geq m$,

$$\left|\sum_{n=m}^{k+1} a_n p^{n!}\right|_p=\left|\sum_{n=m}^{k} a_n p^{n!}+a_{k+1}p^{(k+1)!}\right|_p=\left|\sum_{n=m}^{k} a_n p^{n!}\right|_p.$$

Hence, the infinite sum has the same p-adic valuation as $a_m p^{m!}$ and is not zero. □

Notice that, because of Hensel's lemma, there are in $\mathbb{Z}_p$ integral elements over $\mathbb{Z}$ which are not in $\widehat{\mathbb{Z}}$. In this instance, although the sequence $(p^{n!})$ converges to zero, p is not topologically nilpotent, because $(p^{n!/2})$ does not converge to zero.

Third, $p_1=1$ and $p_n=6^{n!}$ for all $n\geq 2$. We get the completion ring which satisfies $\mathbb{Z}\subset\widehat{\mathbb{Z}}\subset\mathbb{Z}_6$, the last one being the 6-adic integers, which is not an integral domain, but $\widehat{\mathbb{Z}}$ is.

Proposition 6. *If p_n is prime for all $n\geq 2$, then $\widehat{\mathbb{Z}}$ is not a unique factorization domain. The same result holds if $p_n=p^{n!}$ for all $n\geq 2$, with p prime.*

Proof. We only show the first statement. Let $(q_i)_{i\in\mathbb{N}}$ be the sequence of all prime numbers. We construct a nonzero element $\alpha\in\widehat{\mathbb{Z}}$ which is a multiple of all q_i. We pick out a subsequence of $(p_n)_{n\in\mathbb{N}}$, denoted by $(p_{n_i})_{i\in\mathbb{N}}$, so that $p_{n_i}>(q_1q_2\cdots q_{i+1})^{n_i}$, for all $i\in\mathbb{N}$. We define $\alpha=\sum_{i=1}^{\infty} b_i q_1\cdots q_i p_{n_i}$, where the positive integers b_i are to be defined inductively as follows. Firstly $b_1=q_2$ and if $b_1,\dots,b_{m-1}$ have already been selected, then we choose $b_m\in\mathbb{N}$ such that $0<b_m\leq q_{m+1}$ and $\sum_{i=1}^{m} b_i q_1\cdots q_i p_{n_i}$ is a multiple of q_{m+1}. Therefore α is a well defined element of $\widehat{\mathbb{Z}}$ which is a multiple of all q_i. □

3. Completable locally unbounded field topologies on the field of rational numbers

In this section we give a topology on the rational field $\mathbb{Q}$ similar to the one constructed in Section 2. The elements p_n of the sequence $(p_n)_{n\in\mathbb{N}}$ defined in (1) are required to be prime for $n\geq 2$. We define for all $m\in\mathbb{N}$ the following subsets of $\mathbb{Q}$.

$$W_m=\left\{\sum_{n\geq m}^{l}\frac{a_n}{b_n}p_n\colon |a_n|^m,|b_n|^m<p_{n+1},\ b_n \text{ coprime to } p_n\right\}. \tag{15}$$

In the rest of this section, given any sum $\sum_{n=m}^{l}(a_n/b_n)p_n \in W_m$, we will assume that the requirements in (15) are satisfied. In order to prove that this family of subsets is a fundamental system of neighborhoods at zero for a field topology, we need some technical lemmas.

Lemma 7. *If the sequence* $(p_n)_{n\in\mathbb{N}}$ *satisfies the properties* (1), *then*

(a) $\prod_{n=1}^{m}(p_n)^{2^{m-n}} < (p_{m+1})^{3/m}$,
(b) $\prod_{n=1}^{m} p_n \le (p_{m+1})^{1/(m-2)}$, *for all* $m \ge 3$.

Proof. An elementary use of induction. □

Lemma 8. *If* $t \in \mathbb{N}$, $m \ge 5t$ *and* $\sum_{n=m}^{l}(a_n/b_n)p_n \in W_m$, *then*

$$\left(\left(\sum_{n=m}^{l}\left|\frac{a_n}{b_n}\right|p_n\right)\prod_{n=m}^{l}|b_n|\right)^t < p_{l+1}.$$

Proof. Let us consider the product of the denominators

$$\prod_{n=m}^{l}|b_n| \le \prod_{n=m}^{l}(p_{n+1})^{1/m} = \left(\prod_{n=m+1}^{l} p_n\right)^{1/m} p_{l+1}^{1/m}.$$

Lemma 7(b) provides a bound for the expression in the brackets; therefore,

$$\prod_{n=m}^{l}|b_n| \le (p_{l+1})^{(1/(l-2)m)+1/m}.$$

The inequality $l^l \le p_{l+1}$ holds for $l \ge 3$. Whence

$$\left(\sum_{n=m}^{l}\left|\frac{a_n}{b_n}\right|p_n\right)\prod_{n=m}^{l}|b_n| \le \left(\sum_{n=m}^{l}|a_n|p_n\right)\prod_{n=m}^{l}|b_n| \le l\,(p_{l+1})^{1/l}\,p_l\,(p_{l+1})^{\frac{1}{(l-2)m}+\frac{1}{m}}$$

$$\le (p_{l+1})^{1/l+1/l+1/l+\frac{1}{(l-2)m}+\frac{1}{m}} < (p_{l+1})^{1/t}. \qquad \square$$

Lemma 9. *For all* $m \ge 15$, *the elements of* W_m *are represented uniquely in the following sense. If*

$$\sum_{n\ge m}\frac{a_n}{b_n}p_n = \sum_{n\ge m}\frac{c_n}{d_n}p_n \in W_m,$$

with a_n, b_n, c_n, d_n *satisfying conditions* (15), *then* $a_n/b_n = c_n/d_n$, *for all* $n \ge m$.

Proof. Suppose that $a \in W_m$ can be represented in two ways satisfying conditions (15):

$$a = \sum_{n=m}^{l}\frac{a_n}{b_n}p_n = \sum_{n=m}^{l}\frac{c_n}{d_n}p_n \in W_m.$$

We assume that $a_l/b_l \neq c_l/d_l$; then

$$0=\sum_{n=m}^{l} \frac{a_n d_n - c_n b_n}{b_n d_n} p_n.$$

Clearly, we have $|a_n d_n - c_n b_n|^5 < p_{n+1}$ and $|b_n d_n|^5 < p_{n+1}$ for $n=m,\dots,l$. We denote $e_n = a_n d_n - c_n b_n$ and $f_n = b_n d_n$, and we think of $\sum_{n=m}^{l}(e_n/f_n)p_n$ as an element of W_5. Reducing to common denominator, we get

$$\frac{\prod_{n=m}^{l} f_n \left(\sum_{n=m}^{l} \frac{e_n}{f_n} p_n\right)}{\prod_{n=m}^{l} f_n}=0. \tag{16}$$

The numerator splits into the sum

$$\prod_{n=m}^{l} f_n \left(\sum_{n=m}^{l-1} \frac{e_n}{f_n} p_n\right) + \left(\prod_{n=m}^{l-1} f_n\right) e_l p_l. \tag{17}$$

Considering Lemma 8, we have

$$\left|\prod_{n=m}^{l-1} f_n \left(\sum_{n=m}^{l-1} \frac{e_n}{f_n} p_n\right)\right| < p_l;$$

and since f_l and p_l are relatively prime, we obtain that the first term of the sum (17) is relatively prime to p_l. Since the second term is nonzero and divisible by p_l, the numerator of (16) is nonzero and we have a contradiction. □

Lemma 10. *Let $m \geq 15$ and let*

$$\sum_{n\geq m}^{l} \frac{a_n}{b_n} p_n$$

be an element of W_{15} with a_n, b_n satisfying conditions (15). *If $e \in \mathbb{Z}\setminus\{0\}$ and $f \in \mathbb{N}$ satisfy $|e|^3$, $f^3 < p_m$, then*

$$\frac{e}{f} \neq \sum_{n\geq m}^{l} \frac{a_n}{b_n} p_n.$$

In particular, $e/f \notin W_m$.

Proof. We reason by contradiction. Suppose that

$$\frac{e}{f} = \sum_{n\geq m}^{l} \frac{a_n}{b_n} p_n \in W_{15}. \tag{18}$$

If there is only one summand on the right-hand side of this equality, then

$$\frac{e}{f} = \frac{a_l}{b_l} p_l.$$

But this is impossible, since p_l and b_l are coprime and $p_l > |e|^3$. Now, we assume that there are more than one summand in (18) and $a_l \neq 0$. It is clear that

$$\sum_{n\geq m}^{l-1} \frac{a_n}{b_n} p_n \in W_{15}$$

also. We can group this element into a single fraction

$$\frac{c}{d} = \sum_{n\geq m}^{l-1} \frac{a_n}{b_n} p_n.$$

Using Lemma 8, we conclude that $|c|^3, |d|^3 \leq p_l$. We convert the equality (18) into the following one:

$$\frac{a_l}{b_l} p_l = \frac{e}{f} - \frac{c}{d} = \frac{ed - cf}{fd}.$$

But, considering that $|ed - cf| < p_l$ and $(p_l, b_l) = 1$, we reach again a contradiction. □

Theorem 5. *The family $\mathscr{B}_2 = \{W_m\}_{m\in\mathbb{N}}$ is a fundamental system of neighborhoods of zero for a field topology on $\mathbb{Q}$.*

Proof. We have to see that the family $\mathscr{B}_2$ satisfies properties (4)–(9). It is easy to check that (4), (5) and (7) hold. For property (6) one needs Lemma 8; the product of two elements in the neighborhood W_k must be written analogously as we did in (10). Property (8) is a consequence of Lemma 10. The proof of (9), i.e., the continuity of the inverse map, is as follows. Given a neighborhood $W_s \in \mathscr{B}_2$, in view of Lemma 8, there exists $m \geq 15s$ such that each element $a = \sum_{n=m}^{l} (a_n/b_n) p_n \in W_m$ satisfies the inequality

$$\left| \prod_{n=m}^{l} b_n \left(1 + \sum_{n=m}^{l} \left| \frac{a_n}{b_n} \right| p_n \right) \right|^{3s} < p_{l+1}. \tag{19}$$

Let us show that $(1 + W_m)^{-1} \subseteq 1 + W_s$. Given $a = \sum_{n=m}^{l} (a_n/b_n) p_n \in W_m$, we construct the inverse $(1 + a)^{-1} = 1 + \sum_{n=m}^{l} (c_n/b_n) p_n$, where c_n, d_n are determined inductively so that $(1 + a)^{-1} \in 1 + W_s$. For the equality

$$\left(1 + \sum_{n=m}^{l} \frac{a_n}{b_n} p_n \right) \left(1 + \sum_{n=m}^{l} \frac{c_n}{b_n} p_n \right) = 1;$$

to hold, it is sufficient that the equations

$$\left(\frac{a_m}{b_m} + \frac{c_m}{d_m} + \frac{a_m}{b_m}\frac{c_m}{d_m} p_m \right) p_m = 0, \tag{20}$$

$$\left(1 + \sum_{n=m}^{j} \frac{a_n}{b_n} p_n \right) \frac{c_j}{d_j} p_j + \frac{a_j}{b_j} p_j \left(1 + \sum_{n=m}^{j-1} \frac{c_n}{d_n} p_n \right) = 0, \quad j = m+1, \ldots, l \tag{21}$$

are satisfied. From (20), we get

$$\frac{c_m}{d_m} = \frac{-a_m}{b_m + a_m p_m};$$

it is clear that $|c_m|^s$, $|d_m|^s < p_{m+1}$. We now proceed inductively. If we have determined c_n and d_n for $n = m, \dots, j-1$, let us find out c_j and d_j. From (21), we deduce that

$$\frac{c_j}{d_j} = \frac{-a_j \left(1 + \sum_{n=m}^{j-1} \frac{c_n}{d_n} p_n\right) \prod_{n=m}^{j-1} (b_n d_n)}{b_j \left(1 + \sum_{n=m}^{j} \frac{a_n}{b_n} p_n\right) \prod_{n=m}^{j-1} (b_n d_n)}.$$

If $a_j = 0$, we set $c_j = 0$ and $d_j = 1$. We assume that $a_j \neq 0$. Applying the induction hypothesis and Lemma 7(b), we have

$$\prod_{n=m}^{j-1} d_n \leq \prod_{n=m+1}^{j} p_n^{1/s} \leq (p_{j+1})^{1/s(j-2)}.$$

This result together with (19) implies that the denominator d_j satisfies

$$|d_j|^s < (p_{j+1})^{1/3} (p_{j+1})^{1/(j-2)} < p_{j+1}.$$

One can get an analogous bound for the numerator. The denominator d_j splits into the sum

$$b_j \left(1 + \sum_{n=m}^{j-1} \frac{a_n}{b_n} p_n\right) \prod_{n=m}^{j-1} (b_n d_n) + a_j p_j \prod_{n=m}^{j-1} b_n d_n.$$

Since the left-hand term of this sum is relatively prime to p_j, the denominator d_j is too. Thus property (9) is proved. □

We shall denote this topology by $\mathscr{T}_2$. It is a consequence from Lemma 9 that the topology $\mathscr{T}$, defined in Section 2, is finer than the topology $\mathscr{T}_2$ restricted to $\mathbb{Z}$. The topology $\mathscr{T}$ is, indeed, strictly finer than $\mathscr{T}_2|_{\mathbb{Z}}$. Let us consider the sequence $(c_m)_{m \geq 3}$, where

$$c_m = \tfrac{1}{2} p_m + \tfrac{1}{2} p_{m+1}. \tag{22}$$

It converges to zero in $\mathscr{T}_2|_{\mathbb{Z}}$, but is not in V_{15}, since a representation of c_m in V_{15} (see (2)) would be also a representation in W_{15} distinct from (22). Since every neighborhood $W_m \cap \mathbb{Z}$ contains infine primes, $\mathscr{T}_2|_{\mathbb{Z}}$ is not an ideal topology, i.e., it does not have a fundamental system of neighborhoods at zero consisting of ideals. Considering [5, Lemma 11], we deduce the following result.

Proposition 11. *The topological field* $(\mathbb{Q}, \mathscr{T}_2)$ *is not locally bounded.*

The proposition above can also be deduced from the classification of the locally bounded ring topologies on $\mathbb{Q}$ given in [8, 9, 12, p. 392 or 14, p. 195]. The topological field $(\mathbb{Q}, \mathscr{T}_2)$ is not complete. In the next theorem we describe its completion, for which we need the following result.

Lemma 12. *Let $(\alpha_m)_{m\in\mathbb{N}}$ be a Cauchy sequence in $(\mathbb{Q}, \mathscr{T}_2)$. There exist two sequences of integers $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, with b_n, p_n relatively prime, and a Cauchy sequence $(\omega_m)_{m\in\mathbb{N}}$ which is equivalent to $(\alpha_m)_{m\in\mathbb{N}}$ and satisfies $\omega_m = \sum_{n=1}^{m}(a_n/b_n)p_n$, for all m. In addition, $|a_n|^{31}, |b_n|^{31} < p_{n+1}$, for all $n \geq 31$.*

Proof. There exists a subsequence $(\gamma_m)_{m\in\mathbb{N}}$ of $(\alpha_m)_{m\in\mathbb{N}}$ which satisfies $\gamma_{m_1} - \gamma_{m_2} \in W_{m+1}$, for all $m_1, m_2 \geq m$. By Lemma 9, there is unique representation of the elements in W_m, for $m \geq 15$. Besides, for $m \geq 31$, the inclusion $W_{31} + W_m \subseteq W_{15}$ holds. For all $m \geq 31$, the difference $\gamma_m - \gamma_{31}$ belongs to W_{31}; then we can write

$$\gamma_m = \gamma_{31} + \sum_{n=31}^{\mu_m} \frac{a_{mn}}{b_{mn}} p_n, \tag{23}$$

with $|a_{mn}|^{31}, |b_{mn}|^{31} < p_{n+1}$, b_{mn} relatively prime to p_n and $\mu_m \geq m$. We set $a_1/b_1 = \gamma_{31}$ and, for $m \leq 31$, we define $\omega_m = \gamma_{31}$. For $m > 31$, we define

$$\omega_m = \gamma_{31} + \sum_{n=31}^{m} \frac{a_{mn}}{b_{mn}} p_n.$$

Let us show that $(\omega_m)_{m\in\mathbb{N}}$ is a Cauchy sequence equivalent to $(\gamma_m)_{m\in\mathbb{N}}$. Given a neighborhood of zero W_{m+1} in the basis $\mathscr{B}_2$ with $m \geq 31$, let us see that $\gamma_k - \omega_k \in W_m$ for all $k > \mu_m$ (μ_m defined in (23)). Since $\gamma_k - \gamma_m \in W_{m+1}$, we can write

$$\gamma_k - \gamma_m = \sum_{n=m+1}^{\kappa} \frac{u_n}{v_n} p_n,$$

with u_n, v_n satisfying conditions (15). Then

$$\begin{aligned}\gamma_k &= \gamma_{31} + \sum_{n=31}^{\mu_m} \frac{a_{mn}}{b_{mn}} p_n + \sum_{n=m+1}^{\kappa} \frac{u_n}{v_n} p_n \\ &= \gamma_{31} + \sum_{n=31}^{m} \frac{a_{mn}}{b_{mn}} p_n + \sum_{n=m+1}^{\mu_m} \left(\frac{a_{mn}}{b_{mn}} + \frac{u_n}{v_n}\right) p_n + \sum_{n=\mu_m+1}^{\kappa} \frac{u_n}{v_n} p_n.\end{aligned}$$

(We consider $\mu_m > m$; if $\mu_m = m$ the proof follows analogously). Since

$$\sum_{n=31}^{\mu_m} \frac{a_{mn}}{b_{mn}} p_n + \sum_{n=m+1}^{\kappa} \frac{u_n}{v_n} p_n \in W_{31} + W_m \subseteq W_{15},$$

by the uniqueness of the representation of elements in W_{15}, we conclude that

$$\frac{a_{mn}}{b_{mn}} = \frac{a_{kn}}{b_{kn}} \quad \text{for all } n \in \{31,\dots,m\},$$

$$\frac{a_{mn}}{b_{mn}} + \frac{u_n}{v_n} = \frac{a_{kn}}{b_{kn}} \quad \text{for all } n \in \{m+1,\dots,\mu_m\},$$

$$\frac{u_n}{v_n} = \frac{a_{kn}}{b_{kn}} \quad \text{for all } n \in \{\mu_m+1,\dots,\kappa\}.$$

Therefore, assuming $\kappa > k$,

$$\gamma_k - \omega_k = \sum_{n=k+1}^{\kappa} \frac{u_n}{v_n} p_n \in W_{m+1}.$$

The lemma is proved. □

Theorem 6. *The completion of $\mathbb{Q}$ with respect to the field topology $\mathcal{T}_2$ can be written as*

$$\widehat{\mathbb{Q}} = \left\{ \sum_{n=1}^{+\infty} \frac{a_n}{b_n} p_n : \lim_{n\to\infty} \frac{\log(\max\{|a_n|,|b_n|\})}{\log(p_{n+1})} = 0,\ b_n \text{ coprime to } p_n \right\}. \tag{24}$$

Proof. Let $\alpha \in \widehat{\mathbb{Q}}$. Applying the lemma above, there is a Cauchy sequence

$$(\omega_m)_{m\in\mathbb{N}} = \left(\sum_{n=1}^{m} \frac{a_n}{b_n} p_n \right)_{m\in\mathbb{N}},$$

where $a_n, b_n \in \mathbb{Z}$ and b_n, p_n relatively prime, whose limit is α. Then

$$\alpha = \sum_{n=1}^{\infty} \frac{a_n}{b_n} p_n.$$

Given a neighborhood W_k in the basis $\mathcal{B}_2$ with $k \geq 31$, there is $m_k \geq 31$ such that for all $m_2 \geq m_1 \geq m_k$ we have

$$\omega_{m_2} - \omega_{m_1} = \sum_{m=m_1}^{m_2} \frac{a_n}{b_n} p_n \in W_k.$$

Considering that $|a_n|^{31}, |b_n|^{31} < p_{n+1}$ and the uniqueness of the representation of elements in W_k for $k \geq 15$, we conclude that $|a_n|^k, |b_n|^k \leq p_{n+1}$ for all $n > m_k$, and

$$\frac{\log(\max(|a_n|,|b_n|))}{\log(p_{n+1})} \leq \frac{1}{k}.$$

Therefore, the limit of the left-hand side is zero. The converse is similarly established. □

In what follows, given any sum $\sum_{n=1}^{\infty}(a_n/b_n)p_n \in \widehat{\mathbb{Q}}$, we will assume that the conditions in (24) are satisfied. Notice that in (24) it is equivalent to request b_n relatively prime to p_n for all n except possibly a finite number.

In order to prove that the completion ring $\widehat{\mathbb{Q}}$ is actually a field, we need the following result.

Lemma 13. *Given any element $\sum_{n=1}^{\infty}(a_n/b_n)p_n \in \widehat{\mathbb{Q}}\backslash\mathbb{Q}$ and any integer $t \in \mathbb{N}$, there exists $m_t \in \mathbb{N}$ such that for all $m \geq m_t$ we have*

$$\left(\left(\sum_{n=1}^{m}\left|\frac{a_n}{b_n}\right|p_n\right)\prod_{n=1}^{m}|b_n|\right)^t < p_{m+1}.$$

If $m \geq m_t$ and $\sum_{n=1}^{m}(a_n/b_n)p_n = c_m/d_m$, where c_m and d_m are relatively prime integers, then $|c_m|^t, |d_m|^t < p_{m+1}$.

Proof. It suffices to bound the expression

$$\left(\sum_{n=1}^{m}|a_n|p_n\right)\prod_{n=1}^{m}|b_n| \leq m\alpha_m p_m \prod_{n=1}^{m}|b_n|, \tag{25}$$

where $\alpha_m = \max\{|a_1|,\ldots,|a_m|\}$. Given $t \in \mathbb{N}$, let $r = 6t$. There exists $n_0 \in \mathbb{N}$ such that $|b_n|^r < p_{n+1}$ and $n \leq (p_{n+1})^{1/n}$, for all $n \geq n_0$. There exists $n_1 \in \mathbb{N}$ such that $|\alpha_m|^r < p_{m+1}$ and $\prod_{n=1}^{n_0-1}|b_n| < (p_{m+1})^{1/m}$, for all $m \geq n_1$. Applying Lemma 7(b) we obtain

$$\left(\prod_{n=n_0}^{m-1}|b_n|\right)^r \leq \prod_{n=n_0+1}^{m} p_n \leq (p_{m+1})^{1/(m-2)}.$$

Therefore, for $m \geq \max\{n_0, n_1, r\}$, expression (25) is bounded as follows:

$$\left(m\alpha_m p_m\left(\prod_{n=1}^{n_0-1}|b_n|\right)\left(\prod_{n=n_0}^{m-1}|b_n|\right)|b_m|\right)^r$$

$$< (p_{m+1})^{r/m}p_{m+1}(p_{m+1})^{r/m}(p_{m+1})^{r/m}(p_{m+1})^{1/(m-2)}p_{m+1} \leq p_{m+1}^6.$$

The lemma is proved. $\square$

The previous lemma can be easily generalized to the following result.

Lemma 14. *Let any element $\sum_{n=1}^{\infty}(a_n/b_n)p_n \in \widehat{\mathbb{Q}}\backslash\mathbb{Q}$ and any polynomial $R(X) \in \mathbb{Q}[X]$ be given. For each $t \in \mathbb{N}$ there exists $m_t \in \mathbb{N}$ such that for all $m \geq m_t$ we have*

$$R\left(\sum_{n=1}^{m}\frac{a_n}{b_n}p_n\right) = \frac{c_m}{d_m},$$

where c_m and d_m are integers that satisfy $|c_m|^t, |d_m|^t < p_{m+1}$.

Theorem 7. *The completion* $\widehat{\mathbb{Q}}$ *is a field.*

Proof. Let $\alpha = \sum_{n=1}^{\infty}(a_n/b_n)p_n \in \widehat{\mathbb{Q}}\setminus\mathbb{Q}$ presented as in Theorem 6. We construct its inverse $\beta = \sum_{n=1}^{\infty}(c_n/d_n)p_n$, where $c_n, d_n \in \mathbb{Z}$ are defined inductively. We suppose that $a_1 \neq 0$. We set $c_1 = b_1$ and $d_1 = a_1$. Now suppose we have defined c_n and d_n for all $n < m$. The product $\alpha\beta$ can be expressed as follows:

$$\frac{a_1}{b_1}\frac{c_1}{d_1} + \sum_{m=2}^{\infty}\left(\frac{c_m}{d_m}\sum_{n=1}^{m}\frac{a_n}{b_n}p_n + \frac{a_m}{b_m}\sum_{n=1}^{m-1}\frac{c_n}{d_n}p_n\right)p_m.$$

If the coefficients of p_m in the product $\alpha\beta$ are zero for all $m > 1$, then $\beta = \alpha^{-1}$. Thus, we get

$$\frac{c_m}{d_m} = \frac{-a_m\sum_{n=1}^{m-1}\frac{c_n}{d_n}p_n}{b_m\sum_{n=1}^{m}\frac{a_n}{b_n}p_n} = \frac{-a_m\left(\sum_{n=1}^{m-1}\frac{c_n}{d_n}p_n\right)\prod_{n=1}^{m-1}d_nb_n}{b_m\left(\sum_{n=1}^{m}\frac{a_n}{b_n}p_n\right)\prod_{n=1}^{m-1}d_nb_n}. \tag{26}$$

The numerator and denominator of the right-hand side are integers. If $a_m = 0$, we set $c_m = 0$ and $d_m = 1$. Now, assume $a_m \neq 0$. Given $r_1 \geq 3$, we show that there exists $m_1 \in \mathbb{N}$ such that

$$|d_m|^{r_1} \leq p_{m+1} \quad \text{for all } m \geq m_1. \tag{27}$$

We call $r = 3r_1$. By Lemma 13, there exists $l \geq 6$ such that, for all $m \geq l$, the denominator d_m satisfies

$$|d_m|^r \leq \left(\prod_{n=1}^{m-1}|d_n|\right)^r \left|\left(\sum_{n=1}^{m}\frac{a_n}{b_n}p_n\right)\prod_{n=1}^{m}b_n\right|^r \leq \left(\prod_{n=1}^{m-1}|d_n|\right)^r p_{m+1}. \tag{28}$$

We call $d = (\prod_{n=1}^{l-1}|d_n|)^r$. Applying (28), we inductively get

$$|d_l|^r \leq dp_{l+1},$$

$$|d_{l+1}|^r \leq d^2p_{l+1}p_{l+2},$$

$$\ldots,$$

$$|d_{l+k}|^r \leq d^{2^k}\left(\prod_{n=1}^{k}p_{l+n}^{2^{k-n}}\right)p_{l+k+1}.$$

Using Lemma 7(a), the expression on the right-hand side is bounded by

$$d^{2^k}(p_{l+k+1})^{3/(l+k)}p_{l+k+1} \leq d^{2^k}p_{l+k+1}^2.$$

From these inequalities we conclude that

$$|d_m|^r \leq d^{2^m}p_{m+1}^2 \quad \text{for all } m \geq l.$$

There exists $m_0 \in \mathbb{N}$ such that $d^{2^m} \le p_m$, for all $m \ge m_0$. Consequently,

$$|d_m|^r \le p_{m+1}^3 \quad \text{for all } m \ge \max\{l, m_0\}$$

and (27) is proved. It remains to show that the denominator of (26) is relatively prime to p_m, for m big enough. This is an easy consequence of the inequality

$$\left|\left(\sum_{n=1}^{m-1} \frac{a_n}{b_n} p_n\right) \prod_{n=1}^{m-1} b_n\right| < p_m \quad \text{for all } m \text{ big enough.}$$

One can find an analogous bound for the numerator. Then we have constructed the element $\beta \in \widehat{\mathbb{Q}}$ verifying conditions of Theorem 6. □

The topology $\mathscr{T}_2$ can be extended from $\mathbb{Q}$ to $\widehat{\mathbb{Q}}$, as done in Section 2; we shall denote this extended topology also by $\mathscr{T}_2$. A basis of neighborhoods at zero is $\widehat{\mathscr{B}}_2 = \{\overline{W}_m\}_{m \ge 15}$, where $\overline{W}_m$ is the closure of $W_m \in \mathscr{B}_2$ in $\widehat{\mathbb{Q}}$. More explicitly, we have

$$\overline{W}_m = \left\{ \sum_{n=m}^{\infty} \frac{a_n}{b_n} p_n \in \widehat{\mathbb{Q}} : |a_n|^m, |b_n|^m < p_{n+1},\ b_n \text{ coprime to } p_n \right\}.$$

For any sum $\sum_{n=m}^{\infty} (a_n/b_n) p_n \in \overline{W}_m$, we will understand that the above requirements for the cofficients are satisfied. The ring $\widehat{\mathbb{Z}}$, constructed in Section 2, is included in $\widehat{\mathbb{Q}}$. Nevertheless, $(\widehat{\mathbb{Z}}, \mathscr{T})$ is not a topological subring of $(\widehat{\mathbb{Q}}, \mathscr{T}_2)$; the sequence (c_m) defined in (22) converges to zero in $\mathscr{T}_2$, but not in $\mathscr{T}$. Considering this and the fact that $\overline{V}_m$ is included strictly in $\overline{W}_m \cap \widehat{\mathbb{Z}}$ for all $m \ge 15$, we conclude that the topology $\mathscr{T}$ is strictly finer than $\mathscr{T}_2|_{\widehat{\mathbb{Z}}}$. In the next section we see that $\widehat{\mathbb{Q}}$ is not the quotient field of $\widehat{\mathbb{Z}}$.

In the sequel, given the element $\alpha = \sum_{n=1}^{\infty} (a_n/b_n) p_n \in \widehat{\mathbb{Q}}$, we denote $\alpha_k = \sum_{n=k}^{\infty} (a_n/b_n) p_n$ and $\beta_k = \sum_{n=1}^{k-1} (a_n/b_n) p_n$. The next theorem shows a remarkable property of the completion field $\widehat{\mathbb{Q}}$; we shall need the following result.

Lemma 15. *Given $\alpha = \sum_{n=1}^{\infty} (a_n/b_n) p_n \in \widehat{\mathbb{Q}} \setminus \mathbb{Q}$ and $m \in \mathbb{N}$, for all $t \in \mathbb{N}$ there exists $k_t \ge t$ such that, for all $k \ge k_t$,*

$$(\alpha_k)^m = \sum_{n=k}^{\infty} \frac{c_{kn}}{d_{kn}} p_n \in \overline{W}_t,$$

i.e., $|c_{kn}|^t, |d_{kn}|^t < p_{n+1}$ and d_{kn} relatively prime to p_n, for all $n \ge k$.

Proof. The proof is similar to that of Lemma 4, although it is more cumbersome, since one must take care of denominators. We make use of Lemma 7(b) here. □

Theorem 8. *The rational number field is algebraically closed in $\widehat{\mathbb{Q}}$.*

Proof. Let $\alpha = \sum_{n=1}^{\infty} (a_n/b_n) p_n \in \widehat{\mathbb{Q}} \setminus \mathbb{Q}$. Infinitely many a_n are nonzero. Let $R(X) \in \mathbb{Q}[X]$ be a monic polynomial of degree $m \ge 2$. We claim that $R(\alpha) \neq 0$. As we did in the proof of Theorem 4, we split the polynomial $R(X)$ into the following sum:

$$R(X+Y) = P_0(X) + \sum_{i=1}^{m-1} P_i(X)Y^i + Y^m.$$

For each $k>1$ with $a_k \neq 0$, we write $\alpha = \beta_k + \alpha_k$, where β_k and α_k are as defined before Lemma 15. Therefore,

$$R(\alpha) = R(\beta_k + \alpha_k) = P_0(\beta_k) + \sum_{i=1}^{m} P_i(\beta_k)(\alpha_k)^i := P_0(\beta_k) + v. \tag{29}$$

We claim that, for k big enough, this sum cannot be zero. There exists $k_1 \in \mathbb{N}$ such that $P_0(\beta_k) \neq 0$ for all $k \geq k_1$. Given a natural number $t \geq 15$, by Lemma 14, there exists $k_t \in \mathbb{N}$ such that for all $k \geq k_t$ and $i = 0, \dots, m-1$, $P_i(\beta_k) = e_i/f_i$, for some $e_i, f_i \in \mathbb{Z}$ with $|e_i|^{mt}, |f_i|^{mt} < p_k$. There exists k_0 such that $m^t < p_k$, for all $k \geq k_0$. In addition, by Lemma 15, there is $l_t \in \mathbb{N}$ such that for all $k \geq l_t$ and $i = 1, \dots, m$, we have $(\alpha_k)^i = \sum_{n=k}^{\infty} (c_{in}/d_{in}) p_n \in \overline{W}_{3mt}$, where $|c_{in}|^{3mt}, |d_{in}|^{3mt} < p_{n+1}$ and d_{in} coprime to p_n for all $n \geq k$. Therefore, for $k \geq \max\{t, k_0, k_1, k_t, l_t\}$, the second term of the sum in (29) can be written as

$$v = \sum_{n=k}^{\infty} \left(\sum_{i=1}^{m} P_i(\beta_k) \frac{c_{in}}{d_{in}} \right) p_n.$$

In this expression the coefficients of the p_n can be expressed as follows:

$$\sum_{i=1}^{m} \frac{e_i}{f_i} \frac{c_{in}}{d_{in}} = \frac{\prod_{i=1}^{m} (f_i d_{in}) \sum_{i=1}^{m} (e_i/f_i)(c_{in}/d_{in})}{\prod_{i=1}^{m} (f_i d_{in})} = \frac{g_n}{h_n}.$$

Here the numerator is bounded by

$$|g_n| \leq \prod_{i=1}^{m} ((p_k)^{1/mt} (p_{n+1})^{1/3mt}) m (p_k)^{1/mt} (p_{n+1})^{1/3mt}$$

$$\leq (p_k)^{1/t} (p_{n+1})^{1/3t} (p_k)^{1/t} (p_k)^{1/mt} (p_{n+1})^{1/3mt} \leq (p_k)^{3/t} (p_{n+1})^{1/2t} \leq (p_{n+1})^{1/t}.$$

The denominator h_n is bounded analogously. In order to show that h_n is coprime to p_n, it suffices to consider that d_{in} is coprime to p_n and the bound

$$\left| \prod_{i=1}^{m} f_i \right| \leq \prod_{i=1}^{m} (p_k)^{1/mt} = (p_k)^{1/t} < p_n \quad \text{for all } n \geq k.$$

Therefore, $v = \sum_{i=1}^{m} P_i(\beta_k)(\alpha_k)^i = \sum_{n=k}^{\infty} (g_n/h_n) p_n \in \overline{W}_t$. We consider two cases. Firstly, if $v \notin \mathbb{Q}$, then $R(\alpha) = P_0(\beta_k) + v \neq 0$, since $P_0(\beta_k) \in \mathbb{Q}$. Secondly, if $v \in \mathbb{Q}$, then $v = \sum_{n=k}^{l} (g_n/h_n) p_n \in \overline{W}_t$. Since $P_0(\beta_k) = e_0/f_0$, with $|e_0|^{mt}, |f_0|^{mt} < p_k$; applying Lemma 10, we conclude that $P_0(\beta_k) \neq -v$. Thus $R(\alpha) \neq 0$. □

Just as in p-adic analysis, there are nonconstant functions whose derivative is identically zero. Let us give an example. We define a function $f : (\widehat{\mathbb{Z}}, \mathscr{T}) \to (\widehat{\mathbb{Q}}, \mathscr{T}_2)$ as follows; for each $\alpha = \sum_{n=1}^{\infty} a_n p_n \in \widehat{\mathbb{Z}}$ represented according to Lemma 3, $f(\sum_{n=1}^{\infty} a_n p_n) = \sum_{n=1}^{\infty} a_n p_{n+1}$. Fixed $\alpha \in \widehat{\mathbb{Z}}$, there exists a neighborhood of zero $\overline{V}_m \in \widehat{\mathscr{B}}$ such that $f(\alpha + h) = f(\alpha) + f(h)$ for all $h \in \overline{V}_m$. Therefore f is continuous in $\widehat{\mathbb{Z}}$. The derivative $f'(\alpha)$ is, as usual, defined by

$$f'(\alpha) = \lim_{h \to 0} \frac{f(\alpha + h) - f(\alpha)}{h};$$

which in this case equals $\lim_{h \to 0} f(h)/h$. We show that this limit is zero. Given $\overline{W}_l \in \widehat{\mathscr{B}}_2$, there exists $\overline{W}_m \in \widehat{\mathscr{B}}_2$ such that $\overline{W}_m(1 + \overline{W}_m)^{-1} \subseteq \overline{W}_l$. Let $h = \sum_{n=2m}^{\infty} b_n p_n \in \overline{V}_{2m} \setminus \{0\}$, with $|b_{2m}|^{2m} < p_{2m+1}$; we can assume that $b_{2m} \neq 0$ (if not the proof follows similarly). Then

$$\begin{aligned} \frac{f(h)}{h} &= \frac{\sum_{n=2m}^{\infty} b_n p_{n+1}}{\sum_{n=2m}^{\infty} b_n p_n} \\ &= \left(\sum_{n=2m}^{\infty} \frac{b_n}{b_{2m} p_{2m}} p_{n+1} \right) \left(1 + \sum_{n=2m+1}^{\infty} \frac{b_n}{b_{2m} p_{2m}} p_n \right)^{-1} \in \overline{W}_m (1 + \overline{W}_m)^{-1} \subseteq \overline{W}_l. \end{aligned}$$

Consequently, $\lim_{h \to 0} f(h)/h = 0$.

4. Other field topologies on the rational number field

In this section we present other examples of field topologies on the rational field which are similar to the topology in the previous section. All the proofs run analogously to those in the above section and we will omit them. The underlying idea for the topologies in this article is taken from Section 3 of [6]. The following field topology on $\mathbb{Q}$ is even more according to Mutylin's example. Let $(p_n)_{n \in \mathbb{N}}$ be any sequence of natural numbers safisfying conditions (1), (following a suggestion of N. Shell, we do not require that the p_n be prime in this case). For $m \geq 2$ we define the sets

$$\mathscr{V}_m = \left\{ \sum_{n \geq m}^{l} \frac{a_n p_n}{b_n + c_n p_n} : |a_n|^m, |c_n|^m < p_{n+1},\ 0 < |b_n|^m < p_n \right\}.$$

Notice that $\mathscr{V}_{3m} \subset W_m$ if the p_n are primes. The family $\{\mathscr{V}\}_{m \geq 2}$ constitutes a fundamental system of neighborhoods of zero for a Hausdorff field topology on the rational field. We denote this topology by $\mathscr{T}_3$. Now the completion is a field that can be expressed as

$$\mathscr{Q} = \left\{ \sum_{n=1}^{\infty} \frac{a_n p_n}{b_n + c_n p_n} : \begin{array}{l} \lim \dfrac{\log(\max\{|a_n|, |c_n|\})}{\log(p_{n+1})} = \lim \dfrac{\log |b_n|}{\log(p_n)} = 0, \\ \text{and} \ \ b_n \neq 0, \ \text{for all} \ \ n \end{array} \right\}.$$

This field fulfills properties analogous to those of $\widehat{\mathbb{Q}}$ constructed above. In particular, $\mathbb{Q}$ is algebraically closed in $\mathscr{Q}$. Let us point out the following subring of $\mathscr{Q}$.

$$R = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{b_n} p_n : \lim \frac{\log |a_n|}{\log(p_{n+1})} = \lim \frac{\log |b_n|}{\log(p_n)} = 0 \right\}.$$

If $p_1 = 1$ and p_n is prime for $n \geq 2$, we have the subsequent algebraic strict inclusions

$$\widehat{\mathbb{Z}} \subset R \subset \mathscr{Q} \subset \widehat{\mathbb{Q}}.$$

The ring $\widehat{\mathbb{Z}}$ is the one constructed in Section 2. I have not been able to determine whether $\mathscr{Q}$ is the quotient field of R or not.

If we take the sequence $(p_n)_{n\in\mathbb{N}}$ to be $p_1 = 1$ and $p_n = p^{n!}$ for $n \geq 2$, with p prime, we get another field topology on $\mathbb{Q}$ by defining a fundamental system of neighborhoods at zero $\{U_m\}_{m\in\mathbb{N}}$, where

$$U_m = \left\{ \sum_{n\geq m}^{l} \frac{a_n}{b_n} p^{n!-s_n} : |a_n|^m, |b_n|^m < p^{(n+1)!},\ 0 \leq s_n < \frac{n!}{m},\ b_n \text{ coprime to } p \right\}.$$

We denote this topology by $\mathscr{T}_4$. It is finer than the p-adic topology on $\mathbb{Q}$, but coarser than $\mathscr{T}_3$ (assuming $p_n = p^{n!}$ for $n \geq 2$). The completion is a field which can be expressed as

$$\widehat{\mathbb{Q}^p} = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{b_n} p^{n!-s_n} : \begin{array}{l} \lim \dfrac{\log(\max\{|a_n|, |b_n|\})}{(n+1)!} = \lim \dfrac{s_n}{n!} = 0, \\ b_n \text{ coprime to } p \end{array} \right\}.$$

The rational field is also algebraically closed in this field. We now take into consideration the local ring

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \in \mathbb{Q} : b \text{ coprime to } p \right\}.$$

In this ring we define the following sets

$$S_m = \left\{ \sum_{n\geq m}^{l} \frac{a_n}{b_n} p^{n!} : |a_n|^m, |b_n|^m < p^{(n+1)!},\ b_n \text{ coprime to } p \right\}.$$

The family $\{S_m\}_{m\in\mathbb{N}}$ is a fundamental system of neighborhoods at zero for a Hausdorff ring topology on $\mathbb{Z}_{(p)}$. For notational convenience we set $p^{0!} = 1$. The completion is the ring

$$\widehat{\mathbb{Z}_{(p)}} = \left\{ \sum_{n=0}^{\infty} \frac{a_n}{b_n} p^{n!} : \lim \frac{\log(\max\{|a_n|, |b_n|\})}{(n+1)!} = 0,\ b_n \text{ coprime to } p \right\}.$$

This ring is also a topological ring, in which the family $\{\bar{S}_m\}_{m\geq 2}$ is a neighborhood basis at zero, where

$$\bar{S}_m = \left\{ \sum_{n=m}^{\infty} \frac{a_n}{b_n} p^{n!} \in \widehat{\mathbb{Z}_{(p)}} : |a_n|^m, |b_n|^m \leq p^{(n+1)!} \right\}.$$

We denote this topology by $\mathscr{T}_5$. This completion ring is a local domain (possibly not Noetherian) whose maximal ideal is

$$M = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{b_n} p^{n!} : \lim \frac{\log(\max\{|a_n|, |b_n|\})}{(n+1)!} = 0,\ b_n \text{ coprime to } p \right\}.$$

Its quotient field K can be represented as follows.

$$K = \left\{ p^{-m} \sum_{n=0}^{\infty} \frac{a_n}{b_n} p^{n!} : m \in \{0\} \cup \mathbb{N},\ \sum_{n=0}^{\infty} \frac{a_n}{b_n} p^{n!} \in \widehat{\mathbb{Z}_{(p)}} \right\}.$$

The field K can be furnished with a field topology, for which $\mathscr{B}_6 = \{p^l \bar{S}_m\}_{l,m\in\mathbb{N}}$ constitutes a fundamental system of neighborhoods at zero (see [10, Theorem 2, p. 24]). We denote this topology by $\mathscr{T}_6$. I have not been able to get the field K as a completion of $\mathbb{Q}$ for any ring topology. Nevertheless, $(K, \mathscr{T}_6)$ is a complete field, which we prove using an idea from [15, Lemma, Theorem 1].

Proposition 16. *The topological field $(K, \mathscr{T}_6)$ is complete.*

Proof. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(K, \mathscr{T}_6)$. There exists $n_0 \in \mathbb{N}$ such that $\alpha_{n_1} - \alpha_{n_2} \in \bar{S}_3 \subset \widehat{\mathbb{Z}_{(p)}}$ for all $n_1, n_2 \geq n_0$. In particular, $\alpha_n - \alpha_{n_0} \in \widehat{\mathbb{Z}_{(p)}}$ for $n \geq n_0$. Therefore, we can assume that $\alpha_n \in \widehat{\mathbb{Z}_{(p)}}$ for all n. Since the topology $\mathscr{T}_5$ is coarser than $\mathscr{T}_6$ restricted to $\widehat{\mathbb{Z}_{(p)}}$, the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is Cauchy in $(\widehat{\mathbb{Z}_{(p)}}, \mathscr{T}_5)$ too. Thus, it converges to an element $\alpha \in \widehat{\mathbb{Z}_{(p)}}$. Let us show that α is also the limit of $(\alpha_n)_{n\in\mathbb{N}}$ for the topology $\mathscr{T}_6$. We set $\beta_n = \alpha_n - \alpha$ and shall prove that $(\beta_n)_{n\in\mathbb{N}}$ is a null sequence for the topology $\mathscr{T}_6$. Given a neighborhood of zero $p^l \bar{S}_m \in \mathscr{B}_6$, there exists $m_1, r_1 \in \mathbb{N}$ such that $p^l \bar{S}_{m_1} + \bar{S}_r \subseteq \bar{S}_m$, for all $r \geq r_1$. Besides, there exists $n_0 \in \mathbb{N}$ such that $\beta_{n_1} - \beta_{n_2} \in p^l \bar{S}_{m_1}$, for all $n_1, n_2 \geq n_0$. For each $r \geq r_1$ there exists $n_r \geq n_0$ such that $\beta_{n_r} \in \bar{S}_r$. Therefore,

$$\beta_{n_1} = (\beta_{n_1} - \beta_{n_r}) + \beta_{n_r} \in p^l \bar{S}_{m_1} + \bar{S}_r.$$

We conclude that

$$\beta_{n_1} \in \bigcap_{r\in\mathbb{N}} (p^l \bar{S}_{m_1} + \bar{S}_r) \subseteq p^l \bar{S}_m,$$

for all $n_1 \geq n_0$. □

We have the following strict inclusions of rings and fields (taking $p_n = p^{n!}$ for $n \geq 2$):

$$\begin{array}{ccccc} \mathscr{Q} & \subset & \widehat{\mathbb{Q}^p} & \subset & \mathbb{Q}_p. \\ \cup & & \cup & & \\ \widehat{\mathbb{Z}} & \subset & \widehat{\mathbb{Z}_{(p)}} & \subset & K \end{array} \tag{30}$$

The ring $\widehat{\mathbb{Z}}$ is the one constructed in Section 2, and $\mathbb{Q}_p$ the p-adic field. Neither ring nor field in (30) with its corresponding topology is a topological subring of any other in the diagram.

5. Topologies on $k[X]$ and $k(X)$

In this section the same ideas are used to construct a ring topology on $k[X]$ and a field topology on $k(X)$ with similar properties. We shall omit the proofs. In what follows, $Q_n(X)$, $R_n(X)$ will denote polynomials of $k[X]$. We define in $k[X]$ the following subsets:

$$U_m = \left\{ \sum_{n \geq m}^{l} Q_n(X) X^{n!} : \deg(Q_n(X)) < \frac{(n+1)!}{m} \right\}.$$

The family $\{U_m\}_{m \in \mathbb{N}}$ is a fundamental system of neighborhoods of zero for a Hausdorff ring topology on $k[X]$, which we denote by $\mathscr{T}_7$. This topology is finer than the ring topology whose neighborhood basis at zero is the family of ideals $\{(X^n)\}_{n \in \mathbb{N}}$.

Proposition 17. *The topological ring $(k[X], \mathscr{T}_7)$ is not locally bounded.*

Proof. The field k is a bounded set with respect to this ring topology. Obviously, the topology is not an ideal topology. Applying [12, Theorem 37.18] or [1, Theorem 11] we conclude that $(k[X], \mathscr{T}_7)$ is not locally bounded. □

For notational convenience, if $Q_n(X) = 0$, we set $\deg(Q_n(X)) = 0$. The completion of $(k[X], \mathscr{T}_7)$ is the ring

$$\widehat{k[X]} = \left\{ a + \sum_{n=1}^{\infty} Q_n(X) X^{n!} : a \in k \text{ and } \lim_{n \to \infty} \frac{\deg(Q_n(X))}{(n+1)!} = 0 \right\}.$$

This ring is, algebraically, a proper subring of $k[[X]]$. In the field $k(X)$ of rational functions over k we define the following sets

$$W_m = \left\{ \sum_{n=m}^{l} \frac{Q_n(X)}{R_n(X)} X^{n! - s_n} : \begin{array}{l} \deg(Q_n), \deg(R_n) < (n+1)!/m, \\ 0 \leq s_n < n!/m, \ R_n(0) \neq 0 \end{array} \right\}.$$

The family $\{W_m\}_{m\in\mathbb{N}}$ is a neighborhood basis at zero for a field topology on $k(X)$. The completion $\widehat{k(X)}$ is a field constructed analogously to the previous examples. Once again, $k(X)$ is algebraically closed in its completion.

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